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Simple analytical methods for computing the gravity-wave contribution to the cosmic background radiation anisotropy

Yun Wang

NASA/Fermilab Astrophysics Center

Fermi National Accelerator Laboratory, Batavia, IL 60510-0500

Abstract

We present two simple analytical methods for computing the gravity wave contribution to the cosmic background radiation (CBR) anisotropy in inflationary models; one method uses a time-dependent transfer function, the other method uses an approximate gravity-wave mode function which is a simple combination of the lowest order spherical Bessel functions. We compare the CBR anisotropy tensor multipole spectrum computed using our methods with the previous result of the highly accurate numerical method, the "Boltzmann" method. Our time-dependent transfer function is more accurate than the time-independent transfer function found by Turner, White, and Lidsey; however, we find that the transfer function method is only good for $l \lesssim 120$. Using our approximate gravity-wave mode function, we obtain much better accuracy; the tensor multipole spectrum we find differs by less than 2% for $l \lesssim 50$, less than 10% for $l \lesssim 120$, and less than 20% for $l \leq 300$ from the "Boltzmann" result. Our approximate graviton mode function should be quite useful in studying tensor perturbations from inflationary models.

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1. Introduction

The cosmic background radiation (CBR) anisotropy places stringent constraints on theories of the early Universe. Among these theories, the best studied are the inflationary models, which are strongly motivated because they solve the famous problems (the flatness problem, the smoothness problem, the structure formation problem) of the standard cosmology. Tensor (gravity-wave) and scalar (density) metric perturbations are generated in the very early Universe due to quantum fluctuations arising during inflation. Both tensor and scalar perturbations contribute to anisotropy in the temperature of the CBR. While the scalar contribution to the CBR anisotropy involves more complicated physics, the tensor contribution to the CBR anisotropy arises only through the Sachs-Wolfe effect [1] as follows. As photons of the CBR propagate toward us from the last scattering surface, their paths are perturbed by the metric perturbations due to the primordial gravity waves. The perturbed energies of these photons result in temperature fluctuations in the sky that we observe. The CBR temperature fluctuation is conventionally expanded into spherical harmonics:

$$\frac{\delta T}{T}(\theta, \phi) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi) \tag{1.1}$$

In this paper, we present two simple analytic methods of computing the tensor contribution to the variance in the CBR temperature multipole moments, $\langle |a_{lm}|^2 \rangle$, one method makes use of a time-dependent transfer function, the other uses an approximate gravity-wave mode function which is a simple combination of the lowest order spherical Bessel functions. Our methods provide adequate accuracy for normalizing the tensor perturbations arising from inflationary theories to the observable CBR anisotropy. Our method using the approximate gravity-wave mode function is accurate enough to be used in studying the CBR anisotropy tensor multipole spectrum to large l ($l \leq 300$).

We use the results by Dodelson, Knox, and Kolb [2] as the standard for comparison. They considered a Universe with both matter and radiation, and used numerical methods to evolve the photon distribution function using first-order perturbation theory of the general relativistic Boltzmann equation for radiative transfer. Their results (which we refer to as "Boltzmann") should be equivalent to the results obtained using the Sachs-Wolfe formula (see below). The "Boltzmann" method has no simple analytical formulation, and the exact result using the Sachs-Wolfe formula involves complicated spheroidal wavefunctions [3]; hence it is of great interest to find a simple transfer function, or a simple approximate graviton mode function, which can be used analytically to compute the CBR anisotropy tensor multipole moments to sufficient accuracy.

Our first method has been motivated by the tensor transfer function found by Turner, White, and Lidsey (TWL) [4]. Their transfer function takes into account the effect of the Universe becoming matter-dominated gradually. The tensor multipole moments obtained using their transfer function, however, differ from the "Boltzmann" results by up to over 30% for $l < 100$. The reason for this substantial discrepancy is that their transfer function contains no time evolution, although they did give the expression for the time-dependence of the transfer function for short wavelength modes. We modify their transfer function by accounting for the difference in time evolution between long and short wavelength modes.

Our second method has been motivated by intuition. Since the gravity-wave mode function is analytically known for both matter and radiation dominated eras, it should be possible to construct a simple approximate gravity-wave mode function by smooth interpolation. As expected, the resultant mode function is much closer to the true mode function than the much used matter-dominated mode function. When used in computing the CBR anisotropy tensor spectrum, the approximate gravity-wave mode function gives very accurate results.

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2. Gravity wave contribution to CBR anisotropy

The rms temperature fluctuation averaged over the sky for a given experiment is given by

$$\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle = \sum_{l \geq 2} \frac{2l+1}{4\pi} \langle |a_{lm}|^2 \rangle W_l, \tag{2.1}$$

where W_l is the appropriate response function for the experiment. For an experiment with two antennas of Gaussian beam width σ separated by angle θ, the temperature difference between the two antennas is measured; the response function is

$$W_l = 2[1 - P_l(\cos\theta)]\, e^{-(l+1/2)^2 \sigma^2}. \tag{2.2}$$

We have followed the notation of Ref.[4].

Tensor perturbations generated by inflation are stochastic in nature [5]. Let us expand the gravity-wave perturbation in plane waves

$$h_{jk}(\mathbf{x}, \tau) = (2\pi)^{-3} \int \mathrm{d}^3 k \, h_{\mathbf{k}}^i(\tau) \epsilon_{jk}^i \, e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{2.3}$$

where ϵ_{jk}^i is the polarization tensor and $i = \times, +$ in the transverse traceless gauge (in which $h_{00} = h_{0j} = 0$). We have

$$h_{\mathbf{k}}^i(0) = A(k) a^i(\mathbf{k}), \tag{2.4}$$

where $a^i(\mathbf{k})$ is a random variable with statistical expectation value

$$\langle a_i(\mathbf{k})\, a_j(\mathbf{q}) \rangle = k^{-3} \delta^{(3)}(\mathbf{k} - \mathbf{q})\, \delta_{ij}, \tag{2.5}$$

and the spectrum of gravity waves generated by inflation is

$$A^2(k) = \frac{H^2}{\pi^2 M_{\mathrm{PL}}^2} = \frac{8}{3\pi} \frac{V}{M_{\mathrm{PL}}^4}, \tag{2.6}$$

where V is the value of the inflaton potential when the mode with comoving wavenumber k crosses outside the horizon during inflation.

The tensor contribution to the variance of the multipoles is given by

$$\langle |a_{lm}|^2 \rangle = 36\pi^2 \frac{(l+2)!}{(l-2)!} \int \mathrm{d}k \, A^2(k) \, |F_l(k)|^2 \,, \tag{2.7}$$

where

$$F_l(k) = \frac{1}{\sqrt{k}} \int_{\tau_{\mathrm{LSS}}}^{\tau_{\mathrm{now}}} \mathrm{d}\tau \, \frac{\partial}{\partial \tau} \left[\frac{1}{3} \frac{h_{\mathbf{k}}^i(\tau)}{h_{\mathbf{k}}^i(0)} \right] \frac{j_l(k(\tau_{\mathrm{now}} - \tau))}{[k(\tau_{\mathrm{now}} - \tau)]^2}. \tag{2.8}$$

where τ_{now} is the conformal time today, τ_{LSS} is the conformal time at last scattering. We need to find the graviton mode function $h_{\mathbf{k}}^i(\tau)$.

Inflation gives rise to a spatially flat and perturbed Friedmann-Robertson-Walker universe with the metric

$$g_{\mu\nu} = R^2(\tau) \left[\eta_{\mu\nu} + h_{\mu\nu} \right], \tag{2.9}$$

where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$, $h_{\mu\nu}$ is a small perturbation, and τ is the conformal time. The cosmic scale factor $R(\tau)$ is

$$R(\tau) = \left[\tau/\tau_0 + \sqrt{R_{\mathrm{eq}}} \right]^2 - R_{\mathrm{eq}}, \tag{2.10}$$

for a Universe with both matter and radiation. We have defined $\tau_0 \equiv 2H_0^{-1}\sqrt{1 + R_{\mathrm{eq}}}$. At matter-radiation equality, $R(\tau_{\mathrm{eq}}) \equiv R_{\mathrm{eq}} = 4.18 \times 10^{-5} h^{-2}$, and $\tau_{\mathrm{eq}}/\tau_0 = [\sqrt{2} - 1]R_{\mathrm{eq}}^{1/2}$. Today $R_{\mathrm{now}} = 1$, $\tau_{\mathrm{now}}/\tau_0 = \sqrt{1 + R_{\mathrm{eq}}} - \sqrt{R_{\mathrm{eq}}}$. At last scattering, $R_{\mathrm{LSS}} = 1/(1 + z_{\mathrm{LSS}})$, $\tau_{\mathrm{LSS}}/\tau_0 = \sqrt{R_{\mathrm{LSS}} + R_{\mathrm{eq}}} - \sqrt{R_{\mathrm{eq}}}$.

The gravity-wave perturbation satisfies the massless Klein-Gordon equation

$$\ddot{h}_{\mathbf{k}}^i + 2 \left[\frac{\dot{R}}{R} \right] \dot{h}_{\mathbf{k}}^i + k^2 \, h_{\mathbf{k}}^i = 0, \tag{2.11}$$

where $k^2 = \mathbf{k} \cdot \mathbf{k}$ and the overdots denote derivatives with respect to τ.

A gravity-wave mode with wavenumber $\mathbf{k}$ crosses inside the horizon at $k\tau \sim 1$. Before it crosses inside the horizon, $k\tau \ll 1$. Eq.(2.11) gives us $\dot{h}_{\mathbf{k}}^i(\tau) = 0$ for $k\tau \ll 1$, i.e., the gravity-wave mode is frozen before horizon-crossing. We can take $\dot{h}_{\mathbf{k}}^i(0) = 0$ as the

initial condition for Eq.(2.11). For modes that cross inside the horizon during radiation dominated era ($\tau \ll \tau_{\rm eq}$, $R(\tau) = 2\sqrt{R_{\rm eq}}\,\tau/\tau_0$), the exact solution is $h^i_{\mathbf{k}}(\tau) = h^i_{\mathbf{k}}(0)\,j_0(k\tau)$; for modes that cross inside the horizon during matter dominated era ($\tau \gg \tau_{\rm eq}$, $R(\tau) = (\tau/\tau_0)^2$) the exact solution is $h^i_{\mathbf{k}}(\tau) = h^i_{\mathbf{k}}(0)\,3j_1(k\tau)/(k\tau)$. Here $j_0(z) = \sin z/z$ and $j_1(z) = \sin z/z^2 - \cos z/z$ are spherical Bessel functions of order zero and one respectively.

3. First method: time-dependent transfer function

Since the contributions to the tensor multipole moments are dominated by gravity waves which have entered the horizon recently [6], let us write [4]

$$h^i_{\mathbf{k}}(\tau) = h^i_{\mathbf{k}}(0)T_\tau(k/k_{\rm eq})\left[\frac{3j_1(k\tau)}{k\tau}\right], \tag{3.1}$$

where $T_\tau(k/k_{\rm eq})$ is the amplitude transfer function which accounts for the effect of short-wavelength modes entering the horizon during radiation-dominated era, and $k_{\rm eq} \equiv \tau_{\rm eq}^{-1}$. Eq.(2.8) becomes

$$F_l(k) = -k^{3/2}\int_{\tau_{\rm LSS}}^{\tau_{\rm now}}\mathrm{d}\tau\,\tau\,T_\tau(k)\frac{j_2(k\tau)}{(k\tau)^2}\frac{j_l(k(\tau_{\rm now}-\tau))}{[k(\tau_{\rm now}-\tau)]^2}. \tag{3.2}$$

Note that there is no $\partial T_\tau(k)/\partial\tau$ term in the above expression, because $\partial h^i_{\mathbf{k}}(\tau)/\partial\tau$ and $h^i_{\mathbf{k}}(\tau)$ are related to $3j_1(k\tau)/(k\tau)$ and $\partial[3j_1(k\tau)/(k\tau)]/\partial\tau$ by the same amplitude transfer function $T_\tau(k)$.

The transfer function at time τ can be found by integrating Eq.(2.11) numerically from $\tau = 0$ to τ. Today's transfer function is [4]

$$T_{\tau_0}(y) \equiv T_0(y) = \left[1.0 + 1.34y + 2.5y^2\right]^{1/2}, \tag{3.3}$$

where $y \equiv k/k_{\rm eq}$.

Since the Universe became matter-dominated *gradually*, the transfer function in Eq.(3.1) should obviously depend on time. Once a mode is well inside the horizon

$(k\tau \gg 1)$, $h^i_{\mathbf{k}}(\tau) \propto \cos(k\tau)/R$ (see Eq.(2.11)). Since $3j_1(k\tau)/(k\tau)$ is the exact mode function for $R(\tau) = (\tau/\tau_0)^2$, the transfer function for modes with $k \gg k_{\mathrm{eq}}$ at an early time τ is given by [4]

$$T^0_\tau(k/k_{\mathrm{eq}}) = \frac{(\tau/\tau_0)^2}{R(\tau)} T_0(k/k_{\mathrm{eq}}) \equiv A(\tau)\, T_0(k/k_{\mathrm{eq}}), \qquad\qquad k \gg k_{\mathrm{eq}}. \qquad (3.4)$$

The above formula is in very good agreement with numerical results for $k \gtrsim k_{\mathrm{eq}}$ and $\tau_{\mathrm{LSS}} \leq \tau \leq \tau_0$.

On the other hand, modes with $k \ll k_{\mathrm{eq}}$ entered the horizon during matter-dominated era; the transfer function for these modes should have negligible time dependence. Let us write

$$T_\tau(k/k_{\mathrm{eq}}) = T_0(k/k_{\mathrm{eq}})\, T_1(\tau, k/k_{\mathrm{eq}}), \qquad (3.5)$$

where $T_0(k/k_{\mathrm{eq}})$ is given by Eq.(3.3), and $T_1(\tau, k/k_{\mathrm{eq}})$ can be written as

$$T_1(\tau, k/k_{\mathrm{eq}}) = w(k/k_{\mathrm{eq}})A(\tau) + [1 - w(k/k_{\mathrm{eq}})]. \qquad (3.6)$$

where $A(\tau) \equiv (\tau/\tau_0)^2/R(\tau)$. $w(k/k_{\mathrm{eq}}) \to 0$ for $k \ll k_{\mathrm{eq}}$, and $w(k/k_{\mathrm{eq}}) \to 1$ for $k \gg k_{\mathrm{eq}}$. The simplest choice is

$$w(k/k_{\mathrm{eq}}) = 1 - \exp\left[-\gamma\, (k/k_{\mathrm{eq}})^\Delta\right], \qquad (3.7)$$

where γ and Δ are constants.

Since we use the graviton mode function from matter-dominated era, it is consistent to use $\tau_{\mathrm{LSS}} \simeq \tau_0\sqrt{R_{\mathrm{LSS}}}$ (which is the matter-dominated limit of the correct expression) as the lower limit of integration in Eq.(2.8). With $\gamma = 0.9$ and $\Delta = 0.45$ in Eq.(3.7), the multipole moments computed using our transfer function agrees reasonably well with the "Boltzmann" result for $l \lesssim 50$ (see Fig. 1). To get better result at larger l, we can use a slightly complicated weight function

$$w(k/k_{\mathrm{eq}}) = \left(1 - \exp\left[-\gamma\, (k/k_{\mathrm{eq}})^\Delta\right]\right)\left(1 - \exp\left[-n_c(k/k_{\mathrm{eq}} - y_c)^2\right]\right), \qquad (3.8)$$

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where n_c and y_c are constants. With $\gamma = 0.9$, $\Delta = 0.45$, $n_c = 20$, $y_c = 0.7$, the multipole moments computed using our transfer function agrees reasonably well with the "Boltzmann" result for $l \lesssim 100$. In Figure 1, we plot the CBR anisotropy tensor multipole spectrum computed using the TWL transfer function (dotted line), the transfer functions with weight functions given by Eq.(3.7) (dashed line), and Eq.(3.8) (dot-dashed line). The solid line is the "Boltzmann" result.

It can be argued that using the matter-dominated limit for τ_{LSS} distorts the ionization history of the Universe, since the correct expression $\tau_{\mathrm{LSS}} = \tau_0 \left[\sqrt{R_{\mathrm{LSS}} + R_{\mathrm{eq}}} - \sqrt{R_{\mathrm{eq}}} \right]$ is smaller than the matter-dominated limit $\tau_{\mathrm{LSS}} \simeq \tau_0 \sqrt{R_{\mathrm{LSS}}}$ by a factor of $2/3$. However, using the correct expression for τ_{LSS} increases the multipole moments for an amount which increases from 10% at $l = 20$ to 300% at $l = 100$. The reason for this dramatic effect is that the graviton mode function we use is an extremely bad approximation to the true mode function at the era of last scattering. The contribution to the multipole moment for a given l is dominated by the wavenumber at which $F_l(k)$ (see Eq.(2.8)) peaks; since $F_l(k)$ peaks at $k\tau_0 \sim l$, i.e, $k/k_{\mathrm{eq}} \sim 2.678 h^{-1} l \times 10^{-3}$ [4], larger l multipole moments are dominated by the contribution from larger-wavenumber graviton modes, which entered the horizon at earlier times, when the true graviton mode function deviates greatly from the matter-dominated graviton mode function that we use. Our transfer function can *not* correct for this effect even with time dependence included, because at early times (around the last scattering) the graviton mode function has only a smaller number of oscillations in k, while our transfer function only accounts for the difference in *average amplitude* between the true and the matter-dominated mode functions.

By using the matter-dominated limit for τ_{LSS}, we are effectively truncating the integral over conformal time τ in the expression for the multipole moment; it is not surprising that this enables us to get multipole moments (for $l \lesssim 120$) which are not far off from the "Boltzmann" results, since we are cutting off the τ integral at small τ where the

matter-dominated mode function is most inaccurate. The multipole moments we obtain by using the matter-dominated limit for τ_{LSS} are therefore physically consistent.

The transfer function method is limited to $l \lesssim 120$. For larger l, the phase difference between the matter-dominated graviton mode function and the true mode function becomes important, which results in significant deviation between the multipole spectrum computed using *any* transfer function and the multipole spectrum from the "Boltzmann" method.

4. Second method: approximate gravity-wave mode function

If we want to use the correct expression for τ_{LSS}, we *must* use a new mode function which better approximates the true graviton mode function at small τ than the matter dominated mode function $3j_1(k\tau)/(k\tau)$.

Obviously, one can construct an approximate solution which interpolates smoothly between $j_0(k\tau)$ for $\tau \ll \tau_{\mathrm{eq}}$ and $3j_1(k\tau)/(k\tau)$ for $\tau \gg \tau_{\mathrm{eq}}$. Let us write

$$
\begin{aligned}
\frac{\partial}{\partial \tau}\left[\frac{h_{\mathbf{k}}^i(\tau)}{h_{\mathbf{k}}^i(0)}\right] &= [1 - w(\tau)] \, T_\tau^0(k/k_{\mathrm{eq}}) \frac{\partial}{\partial \tau}\left[\frac{3j_1(k\tau)}{k\tau}\right] + w(\tau) \frac{\partial j_0(k\tau)}{\partial \tau}, \\
&= -k\left\{[1 - w(\tau)] \, T_\tau^0(k/k_{\mathrm{eq}})\left[\frac{3j_2(k\tau)}{k\tau}\right] + w(\tau) \, j_1(k\tau)\right\},
\end{aligned} \tag{4.1}
$$

where $w(\tau) \to 0$ for $\tau \gg \tau_{\mathrm{eq}}$, and $w(\tau) \to 1$ for $\tau \ll \tau_{\mathrm{eq}}$. $T_\tau^0(k/k_{\mathrm{eq}}) = [(\tau/\tau_0)^2/R(\tau)] \, T_0(k/k_{\mathrm{eq}})$ (see Eq.(3.4)), it correctly accounts for the difference in average amplitude between the matter-dominated mode function and the true mode function.

Since $j_0(k\tau)$ accurately gives both the amplitude and the phase of the true mode function at large k, we should "turn off" $T_{\tau_0}(k/k_{\mathrm{eq}})$ (which accounts for the amplitude difference between $3j_1(k\tau)/(k\tau)$ and the true mode function) at large k. We can replace Eq.(3.3) with

$$
T_0(y) = e^{-ay^b}\left[1.0 + 1.34y + 2.5y^2\right]^{1/2} + \left(1 - e^{-ay^b}\right), \qquad y \equiv k/k_{\mathrm{eq}}. \tag{4.2}
$$

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To compute the multipole moments, we make the following substitution in Eq.(3.2):

$$T_\tau(k) \frac{j_2(k\tau)}{k\tau} \Longrightarrow [1 - w(\tau)] \, T_\tau^0 \, (k/k_{\mathrm{eq}}) \, \frac{j_2(k\tau)}{k\tau} + w(\tau) \frac{j_1(k\tau)}{3}. \tag{4.3}$$

A good choice for $w(\tau)$ is

$$w(\tau) = e^{-\alpha(\tau/\tau_{\mathrm{eq}})^\beta}, \tag{4.4}$$

where α and β are constants.

Fitting the resultant tensor multipole spectrum to that found by the "Boltzmann" method, we find $\alpha = 0.2$, $\beta = 0.65$, $a = b = 4$ (the fitting is more sensitive to α and β than to a and b). The agreement between our result with the "Boltzmann" result is impressive: better than 2% for $l \lesssim 50$, better than 10% for $l \lesssim 120$, and better than 20% for $l \leq 300$. In Figure 2, we plot the CBR anisotropy tensor multipole spectrum computed using the matter-dominated graviton mode function with the TWL transfer function (dotted line); our approximate graviton mode function (given by Eqs.(4.1), (4.2), and (4.4)) (dashed line). The solid line is the "Boltzmann" result.

The multipole spectrum computed using the "Boltzmann" method has a peak at $l \simeq 217$ (see Figure 2). The multipole spectrum computed using our approximate mode function has a peak at $l \simeq 213$, while the multipole spectrum computed using the matter-dominated mode function (with or without transfer function) has a peak at $l \simeq 180$ (see Figure 2). Our approximate mode function gives much more accurate phase information than the matter-dominated mode function.

The successful application of our approximate graviton mode function in computing the tensor multipole spectrum stems from the fact that it is rather close to the true graviton mode function. In Figures 3 and 4, we plot the conformal time derivatives of the true mode function (solid line), our approximate mode function (given by Eqs.(4.1), (4.2), and (4.4)) (dashed line), and the matter dominated mode function (dotted line).

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Figure 3 shows that at $\tau = \tau_{\text{LSS}}$, our approximate mode function is much closer to the true mode function than the matter-dominated mode function, for all wavenumbers k. Figure 4 shows that for $k = k_{\text{eq}} \equiv \tau_{\text{eq}}^{-1}$, our approximate mode function is also much closer to the true mode function than the matter-dominated mode function.

5. Generalization

The CBR anisotropy tensor multipole spectrums in Figures 1 and 2 are for the standard values of $h = 0.5$, $\Omega_{\text{B}} = 0.05$, $\Omega_0 = 1$; and for a scale-invariant primordial spectrum of gravity waves. It is straightforward to use our methods to compute the tensor multipole spectrum for a non-scale-invariant primordial spectrum of gravity waves, just use the corresponding $A(k)$ (see Eq.(2.6), which is constant in the scale-invariant case) in Eq.(2.7).

Next, we consider the cases when some or all of the parameters h, Ω_{B}, and Ω_0 are different from the standard values ($h = 0.5$, $\Omega_{\text{B}} = 0.05$, $\Omega_0 = 1$). z_{LSS} is given by [5]

$$1 + z_{\text{LSS}} \simeq 1100 \left(\Omega_0/\Omega_{\text{B}}\right)^{0.018}. \tag{5.1}$$

As long as $\Omega_0 = 1$, the cosmic scale factor is given by Eq.(2.10); our previous formalism applies with z_{LSS} given above. In Figure 5, we plot the CBR anisotropy tensor multipole spectrum computed using our approximate graviton mode function (given by Eqs.(4.1), (4.2), and (4.4)) for $\Omega_0 = 1$, $\Omega_{\text{B}} = 0.05$, $h = 0.8$ (solid line) and $h = 0.5$ (dashed line).

If $\Omega = \Omega_0 + \Omega_\Lambda = 1$, but $\Omega_0 < 1$, we have

$$\frac{\tau}{\tau_0} = \frac{1}{2} \int_0^{R(\tau)} \frac{\mathrm{d}R}{\sqrt{R_{\text{eq}} + R + (\Omega_\Lambda/\Omega_0)(1 + R_{\text{eq}}) R^4}},$$

$$\tau_0 \equiv 2\Omega_0^{-1/2} H_0^{-1} \sqrt{1 + R_{\text{eq}}},$$

$$R_{\text{eq}} = 4.18 \times 10^{-5} (\Omega_0 h^2)^{-1},$$

$$k_{\text{eq}} \equiv \tau_{\text{eq}}^{-1} = \frac{\tau_0^{-1}}{(\sqrt{2} - 1)\sqrt{R_{\text{eq}}}},$$

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$$\frac{\tau_{\mathrm{LSS}}}{\tau_0} = \sqrt{R_{\mathrm{LSS}} + R_{\mathrm{eq}}} - \sqrt{R_{\mathrm{eq}}}. \tag{5.2}$$

Note that the conformal time today $\tau_{\mathrm{now}} \neq \tau_0$. For $h = 0.8$, $\Omega_0 = 0.2$, $\Omega_\Lambda = 0.8$, $\tau_{\mathrm{now}}/\tau_0 \simeq 0.85$. A sizable cosmological constant term significantly alters the late time evolution of the cosmic scale factor, hence $3j_1(k\tau)/(k\tau)$ is no longer the mode function at late times. However, our approximate graviton mode function is still a good approximation of the true mode function in the context of computing tensor multipole spectra, which are not very sensitive to the late time graviton mode function [4]. In Figure 6, we plot the CBR anisotropy tensor multipole spectrum computed using our approximate graviton mode function for $\Omega_0 = 0.2$, $\Omega_\Lambda = 0.8$, $\Omega_{\mathrm{B}} = 0.02$, $h = 0.8$ (solid line), and $\Omega_0 = 1$, $\Omega_{\mathrm{B}} = 0.05$, $h = 0.5$ (dashed line).

We expect the tensor multipole spectrum computed using our approximate graviton mode function (given by Eqs.(4.1), (4.2), and (4.4), with $\alpha = 0.2$, $\beta = 0.65$, $a = b = 4$) to have fairly good accuracy when the cosmological parameters h, Ω_{B}, and Ω_0 differ from the standard values of $h = 0.5$, $\Omega_{\mathrm{B}} = 0.05$, $\Omega_0 = 1$.

6. Discussion

We have presented two simple and straightforward analytical methods for computing the CBR anisotropy tensor multipole spectrum; one method uses a time-dependent transfer function, the other method uses an approximate gravity-wave mode function which is a simple combination of the lowest order spherical Bessel functions (given by Eqs.(4.1), (4.2), and (4.4), with $\alpha = 0.2$, $\beta = 0.65$, $a = b = 4$). Both methods give much better accuracy than using the matter-dominated mode function with the time independent transfer function of Turner, White, and Lidsey [4]. Our approximate graviton mode function method is especially promising, since it gives a tensor multipole spectrum which is impressively accurate (difference with the "Boltzmann result" is less than 2% for $l \lesssim 50$,

less than 10% for $l \lesssim 120$, and less than 20% for $l \leq 300$).

Many attempts have been made in the past in finding a good approximate graviton mode function, for instance, via the sudden approximation (assuming the transition from radiation-domination to matter-domination to be instantaneous) [7]. In addition, Ng and Speliotopoulos explored the possibility of finding a good approximate graviton mode function via the WKB approximation [8]; however, they did not use the mode function they found in computing the multipole moments, they used the mode function found by numerical integration instead. Furthermore, Koranda and Allen found the exact graviton mode function in terms of spheroidal wavefunctions [3]. The advantage of our approximate graviton mode function is that it takes into account the *gradual* transition from radiation-domination to matter-domination in the Universe; it can be used to compute the CBR anisotropy tensor multipole spectrum rather accurately; and it involves only the first and second order spherical Bessel functions.

In a different direction of the CBR anisotropy tensor multipole spectrum calculation, the "Boltzmann method" evolves the photon distribution function numerically, using first-order perturbation theory of the general relativistic Boltzmann equation for radiative transfer [2]. We have used the CBR anisotropy tensor multipole spectrum from the "Boltzmann method" as the standard for comparison, but we do not think that it is practical to use the numerically rather involved "Boltzmann method" in all cases. When we study the tensor perturbations from a large number of inflationary models, it is much more convenient to use the Sachs-Wolfe formula with a simple yet reasonably accurate graviton mode function, such as the one presented in this paper.

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Figure Captions

Figure 1. The CBR anisotropy tensor multipole spectrum computed using various transfer functions, compared to the Boltzmann result (solid line).

Figure 2. The CBR anisotropy tensor multipole spectrum computed using our approximate graviton mode function (given by Eqs.(4.1), (4.2), and (4.4)), compared to the Boltzmann result (solid line).

Figure 3. The conformal-time derivatives at $\tau = \tau_{\mathrm{LSS}}$, of the true graviton mode function (solid line), our approximate mode function (given by Eqs.(4.1), (4.2), and (4.4)) (dashed line), and the matter-dominated mode function (dotted line).

Figure 4. The conformal-time derivatives with $k = k_{\mathrm{eq}} \equiv \tau_{\mathrm{eq}}^{-1}$, of the true graviton mode function (solid line), our approximate mode function (given by Eqs.(4.1), (4.2), and (4.4)) (dashed line), and the matter-dominated mode function (dotted line).

Figure 5. The CBR anisotropy tensor multipole spectrum computed using our approximate graviton mode function (given by Eqs.(4.1), (4.2), and (4.4)) for $\Omega_0 = 1$, $\Omega_{\mathrm{B}} = 0.05$, $h = 0.8$ (solid line) and $h = 0.5$ (dashed line).

Figure 6. The CBR anisotropy tensor multipole spectrum computed using our approximate graviton mode function (given by Eqs.(4.1), (4.2), and (4.4)) for $\Omega_0 = 0.2$, $\Omega_{\Lambda} = 0.8$, $\Omega_{\mathrm{B}} = 0.02$, $h = 0.8$ (solid line) and $\Omega_0 = 1$, $\Omega_{\mathrm{B}} = 0.05$, $h = 0.5$ (dashed line).

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